

Article

Performance Measures in a Generalized Asymmetric Simple Inclusion Process

Yaron Yeger * and Uri Yechiali

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel; uriy@tauex.tau.ac.il

* Correspondence: yaronyeg@mail.tau.ac.il

Abstract: Performance measures are studied for a generalized n -site asymmetric simple inclusion process (G-ASIP), where a general process controls intervals between gate-opening instants. General formulae are obtained for the Laplace–Stieltjes transform, as well as the means, of the (i) traversal time, (ii) busy period, and (iii) draining time. The PGF and mean of (iv) the system’s overall load are calculated, as well as the probability of an empty system, along with (v) the probability that the first occupied site is site k ($k = 1, 2, \dots, n$). Explicit results are derived for the wide family of gamma-distributed gate inter-opening intervals (which span the range between the exponential and the deterministic probability distributions), as well as for the uniform distribution. It is further shown that a homogeneous system, where at gate-opening instants gate j opens with probability $p_j = \frac{1}{n}$, is optimal with regard to (i) minimizing mean traversal time, (ii) minimizing the system’s load, (iii) maximizing the probability of an empty system, (iv) minimizing the mean draining time, and (v) minimizing the load variance. Furthermore, results for these performance measures are derived for a homogeneous G-ASIP in the asymptotic cases of (i) heavy traffic, (ii) large systems, and (iii) balanced systems.

Keywords: asymmetric simple inclusion process (ASIP); generalized ASIP (G-ASIP); performance measures; limit laws



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1. Introduction

A tandem stochastic system (TSS) is an n -site network in a series, where random events cause particles (customers, messages, products, calls, jobs, molecules, etc.) to propagate unidirectionally, under specific stochastic rules, from one site to the next along the one-dimensional array of sites (queues, servers, stations, etc.) until exiting the system. Particles are fed, randomly in time, into the leftmost site and propagate unidirectionally (to the right) through the system. At the rightmost site, particles exit the system randomly in time. The random inflow into the leftmost site, the random instants of movement from site to site, and the random outflow from the rightmost site are all governed by random processes. This model in different variations has been widely explored in the literature. In a TSS, each site has a buffer and a gate in front of it. When a site’s gate opens, particles move to the next site. Varying assumptions on sites’ buffer sizes, as well as on gate (service) capacities, lead to different models of a TSS.

A well-known, fundamental TSS is the tandem Jackson network (TJN), which has been investigated thoroughly in many papers and books (see, e.g., [1–7]). In this TSS model, each site’s buffer size is unlimited. However, the service capacity of each site is one, which allows only single particles to move from site to site, while the transitions follow Markovian rules. The TJN is famous for its product-form solution of the joint multi-dimensional probability generating function (PGF) of the sites’ queue sizes (occupancies).

Another notable TSS is the asymmetric simple exclusion process (ASEP) (see, e.g., [8–11]), where the buffer size of each site is only one, allowing sites to hold at most a single particle at a time. Since the buffer size of each site is limited to one, while the service capacity

of each site might be unlimited, de facto, each site does not service more than a single particle at a time. The exclusion principle of the ASEP model causes blockings of particles' forward movements through the system. The analysis of particle movements under these obstructions is non-trivial and complicated. The ASEP, its various applications, and related models have been further investigated in [12–24].

An important TSS model that bridges the gap between the TJN and the ASEP is the asymmetric simple inclusion process (ASIP), introduced recently by Reuveni, Eliazar, and Yechiali (REY [25]). Similar to the TJN and the ASEP, in the ASIP system, each site has a buffer where particles accumulate. The buffer size (capacity) of each site, as well as its service capacity, is unlimited. In contrast to the TJN and ASEP models, where only a single particle moves forward from site to site, under the ASIP model, when a gate opens (service at a site is completed), all particles accumulated there move simultaneously to the next site.

In all three TSS models, particles arrive at the first site following a Poisson process, while gate j ($j = 1, 2, \dots, n$) opens, independently of other gates, every exponential time at a site-dependent rate. Various performance measures of the system and limit laws were studied in [26].

Boxma, Kella, and Yechiali (BKY) [27] extended the analysis of ASIP systems and investigated a generalized ASIP model where (i) times between gate openings are determined by a renewal process, O , and (ii) particles arrive at all sites during the time interval between two gate openings. Buffer capacities are unlimited, and at the instant of a gate opening, only one gate opens, with gate j opening at a probability $p_j > 0$, $\sum_{j=1}^n p_j = 1$. When gate j opens, all particles residing there move simultaneously to gate $j + 1$. The focus in [27] is on the derivation of the multi-dimensional joint PGF of the site occupancies.

The current paper concentrates on deriving expressions for key performance measures for G-ASIP systems with the particles' arrival at the first site and a general distribution of gate inter-opening times. Five performance measures are investigated: (i) traversal time—the random duration of time it takes a particle to traverse through the system, namely, the time elapsing from the instant a particle is admitted to the first site, until the instant thereafter it exits the system; (ii) overall load—the total number of particles present in the system in steady-state; (iii) busy period—the random time duration in which the system is continuously non-empty; (iv) first occupied site—the first non-empty site in the system; and (v) draining time—the duration of time it takes the network to clear, namely, the random time elapsing from the moment the arrival process stops, until the first instant thereafter that the system becomes empty. Moreover, considering the family of gamma probability distribution functions, denoted $\Gamma(\alpha, \alpha\mu)$, as gate inter-opening times, which spans the range of distributions between the exponential ($\alpha = 1$) and the deterministic ($\alpha \rightarrow \infty$), all have the same mean $E[O] = \frac{\alpha}{\alpha\mu} = \frac{1}{\mu}$, explicit formulae are derived for the Laplace–Stieltjes transforms, PGFs, and means of the above performance measures. In addition, we analyze the system under uniform gate inter-opening times and compare the results to those obtained for the deterministic distribution.

Section 2 is comprised of five subsections in which each one of the above five performance measures is analyzed. Specifically, explicit results are derived for four different gate inter-opening times: (i) gamma distribution, (ii) exponential distribution, (iii) deterministic distribution, and (iv) uniform distribution. Comparisons between the uniform and the deterministic inter-opening times are presented in Sections 2.1–2.3.

In Section 3, the G-ASIP is optimized with respect to various objective functions. It is shown that a homogeneous G-ASIP (where all gates open with equal probability at gate-opening instants) is optimal with respect to (i) minimizing the mean traversal time, (ii) minimizing the mean load system, (iii) maximizing the probability of an empty system, (iv) minimizing the mean draining time, and (v) minimizing the load variance.

In Section 4, a homogeneous G-ASIP is analyzed for the asymptotic cases of: (i) heavy traffic, (ii) large systems, and (iii) balanced systems. The five measures treated in Section 2 are further studied in the above cases, with an emphasis on balanced systems. The Laplace transform, or PGF, of the performance measures, along with their means, are derived for

the above four different gate inter-opening times. We further show that in the intriguing case of balanced systems, the five measures have similar behavior and share the same limiting distributions when gate inter-opening times are: (i) exponential, (ii) deterministic, or (iii) uniform.

2. Performance Measures

Consider an n-site generalized ASIP where a general gate-opening process controls the evolution of the system: the time duration between two consecutive gate-opening instants is a random variable, O , with mean $E[O]$ and Laplace–Stieltjes transform (LST) $\tilde{O}(s)$, while successive intervals between gate openings are i.i.d., all distributed like O . At a gate-opening instant, only one gate opens, where gate j opens with probability $p_j > 0$, $\sum_{j=1}^n p_j = 1$. Arriving particles accumulate in front of the system and are admitted into the first site’s buffer immediately after its first gate opens following their arrival. The number of arrivals between any two consecutive gate openings has a PGF $\hat{A}(z)$, with mean $\hat{A}'(1)$. Let $\lambda = \frac{\hat{A}'(1)}{E[O]}$ be the particles’ rate of arrival.

The five measures defined above are analyzed, each one in a separate Subsection: (i) traversal time (Section 2.1), (ii) overall load (Section 2.2), (iii) busy period (Section 2.3), (iv) first occupied site (Section 2.4), and (v) draining time (Section 2.5). In each subsection, general results are derived for general inter-opening times, followed by explicit formulae for the gamma, exponential, deterministic, and uniform distributions of O . Comparisons between the deterministic and uniform distributions are presented.

2.1. Traversal Time

Let T denote the traversal time of a particle through the system; that is, the time elapsing from the instant a particle is admitted to the first site until it exits the system (after passing through all n sites). Clearly, the traversal time is independent of the arrival process.

Let $\{O_j, j = 1, 2, 3, \dots\}$ be a sequence of i.i.d. random durations between successive gate-opening instants, all distributed like O , while Y_j denotes the number of gate-opening instants between two consecutive openings of gate j , which is geometrically distributed with parameter p_j . That is, $P(Y_j = m) = (1 - p_j)^{m-1} p_j, m = 1, 2, 3, \dots$. Thus, the sojourn time of a particle in site j is $T_j = \sum_{k=1}^{Y_j} O_k$, so that the traversal time is $T = \sum_{j=1}^n T_j$, while T_j ’s are independent. The Laplace–Stieltjes transform (LST) of T is given by

$$\tilde{T}(s) = E[e^{-sT}] = \prod_{j=1}^n \tilde{T}_j(s) = \prod_{j=1}^n E_{Y_j}[\tilde{O}^{Y_j}(s)] = \prod_{j=1}^n \sum_{m=1}^{\infty} [\tilde{O}^m(s)(1 - p_j)^{m-1} p_j] = \prod_{j=1}^n \frac{p_j \tilde{O}(s)}{1 - (1 - p_j) \tilde{O}(s)}, \tag{1}$$

while its mean is

$$E[T] = -\tilde{T}'(0) = E[O] \sum_{j=1}^n \frac{1}{p_j}. \tag{2}$$

Indeed, as Y_j and O are independent, the mean sojourn time of a particle in site j is

$$E[T_j] = E[Y_j] \cdot E[O] = \frac{1}{p_j} E[O].$$

2.1.1. Gamma-Distributed Inter-Opening Intervals

Suppose that the inter-opening time O is distributed like $\Gamma(\alpha, \alpha\mu)$, with density $h(t) = e^{-\alpha\mu t} \frac{(\alpha\mu)^\alpha t^{\alpha-1}}{\Gamma(\alpha)}$, mean $E[O] = \frac{\alpha}{\alpha\mu} = \frac{1}{\mu}$, and LST $\tilde{\Gamma}(\alpha, \alpha\mu)(s) = \left(\frac{\alpha\mu}{s+\alpha\mu}\right)^\alpha$, where $\Gamma(\alpha) = \int_0^\infty e^{-\alpha\mu t} (\alpha\mu)^\alpha t^{\alpha-1} dt$. Then, Equation (1) becomes

$$\tilde{T}(s) = \prod_{j=1}^n \frac{p_j \left(\frac{\alpha\mu}{s+\alpha\mu}\right)^\alpha}{1 - (1-p_j) \left(\frac{\alpha\mu}{s+\alpha\mu}\right)^\alpha} = \prod_{j=1}^n \frac{p_j}{\left(1 + \frac{s}{\alpha\mu}\right)^\alpha - (1-p_j)} \tag{3}$$

Clearly, $E[T] = \frac{1}{\mu} \sum_{j=1}^n \frac{1}{p_j}$.

Special case 1: Exponential inter-opening intervals

When $\alpha = 1$, since the geometric (p_j) sum of i.i.d exponentials (μ) is exponential with parameter $p_j\mu$, it follows that gate j opens independently every exponential time with parameter $\mu_j = p_j\mu$. Thus, Equation (3) coincides with Equation (4) in [26], namely,

$$\tilde{T}(s) = \prod_{j=1}^n \frac{p_j\mu}{p_j\mu + s} = \prod_{j=1}^n \frac{\mu_j}{\mu_j + s} \tag{4}$$

Clearly, $E[T] = \sum_{j=1}^n \frac{1}{\mu_j}$, since a traversing particle sojourns in each site an exponential time with mean $\frac{1}{\mu_j}$.

Special case 2: Deterministic inter-opening intervals

When $\alpha \rightarrow \infty$, the $\Gamma(\alpha, \alpha\mu)$ probability distribution becomes a deterministic inter-opening interval. Then,

$$\lim_{\alpha \rightarrow \infty} \tilde{T}(s) = \prod_{j=1}^n \frac{p_j}{e^{\frac{s}{\mu}} - (1-p_j)} \tag{5}$$

2.1.2. Uniformly Distributed Inter-Opening Intervals

Suppose that the inter-opening time O is uniformly distributed like $U\left[0, \frac{2}{\mu}\right]$, with density $h(t) = \frac{\mu}{2}$, mean $E[O] = \frac{1}{\mu}$, and LST $\tilde{O}(s) = \frac{\mu}{2s} \left(1 - e^{-\frac{2s}{\mu}}\right)$.

Then, Equation (1) becomes

$$\tilde{T}(s) = \prod_{j=1}^n \frac{p_j \frac{\mu}{2s} \left(1 - e^{-\frac{2s}{\mu}}\right)}{1 - (1-p_j) \frac{\mu}{2s} \left(1 - e^{-\frac{2s}{\mu}}\right)} = \prod_{j=1}^n \frac{p_j}{\frac{2s}{\mu} \left(1 - e^{-\frac{2s}{\mu}}\right)^{-1} - (1-p_j)} \tag{6}$$

while the mean $E[T]$ is given again by

$$E[T] = E[O] \cdot \sum_{j=1}^n \frac{1}{p_j} = \frac{1}{\mu} \sum_{j=1}^n \frac{1}{p_j}$$

2.1.3. Comparison: Deterministic vs. Uniform

Comparing the deterministic PDF against the uniform PDF shows that, although both have the same mean traversal time, examining Equations (5) and (6) shows that $\frac{2s}{\mu} \left(1 - e^{-\frac{2s}{\mu}}\right)^{-1} \leq e^{\frac{s}{\mu}}$. That is, the value of the Laplace–Stieltjes transform of the traversal time under the uniform inter-opening times is higher than its corresponding value under

the deterministic distribution. This readily follows since the variance of the uniform distribution is higher than the zero variance of the deterministic distribution.

2.2. Overall Load and the Probability of an Empty System

Consider the system in steady-state. Let X_j denote the number of particles (occupancy, or load) residing in site j right before a gate opening and let $L_k = \sum_{j=1}^k X_j$ denote the total load of the first k sites, $k = 1, 2, \dots, n$. The overall load of the entire system is $L = L_n$. It was shown in Equation (24) in [27] that the PGF of L is given by

$$G_L(z) = E[z^L] = \prod_{j=1}^n \frac{\hat{A}(z)p_j}{1 - \hat{A}(z)(1 - p_j)}, \tag{7}$$

implying that

$$E[L] = \hat{A}'(1) \sum_{j=1}^n \frac{1}{p_j} = \lambda \cdot E[O] \cdot \sum_{j=1}^n \frac{1}{p_j} = \lambda \cdot E[T],$$

which is, in fact, Little’s Law.

From Equation (7), the probability that the system is empty is given by

$$P(L = 0) = G_L(0) = \prod_{j=1}^n \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)}, \tag{8}$$

where $\hat{A}(0)$ is the probability of no arrivals during the inter-opening time O .

2.2.1. Gamma-Distributed Inter-Opening Intervals

As in previous ASIP studies, we assume that the arrival process is independent of the inter-gate opening distribution and is Poisson with rate λ . Then,

$$\hat{A}(z) = \left(\frac{\alpha\mu}{\lambda(1-z) + \alpha\mu} \right)^\alpha \quad \text{and} \quad \hat{A}(0) = \left(\frac{\alpha\mu}{\lambda + \alpha\mu} \right)^\alpha. \tag{9}$$

Consequently, substituting Equation (9) in Equation (7) results in

$$E[z^L] = \prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda(1-z)}{\alpha\mu}\right)^\alpha - (1 - p_j)}, \tag{10}$$

and

$$P(L = 0) = \prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - (1 - p_j)}. \tag{11}$$

Note that $P(L = 0)$ is a strictly decreasing function of α , since $\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha$ is a strictly increasing function of α for a given ratio of $\frac{\lambda}{\mu}$. However, the decreasing rate is mild. For example, consider a homogeneous system with $n = 10$ sites, where $p_j = \frac{1}{n} = \frac{1}{10}$, $\lambda = \frac{1}{10}$, and $\mu = 4$. Then,

$$P(L = 0)(\alpha) = \prod_{j=1}^{10} \frac{1}{10 \left(1 + \frac{1}{40\alpha}\right)^\alpha - 9}.$$

Thus, when $\alpha = 1$

$$P(L = 0)(1) = \prod_{j=1}^{10} \frac{1}{10 \left(1 + \frac{1}{40}\right)^\alpha - 9} = 0.107,$$

while when α becomes large

$$\lim_{\alpha \rightarrow \infty} P(L = 0)(\alpha) = \prod_{j=1}^{10} \frac{1}{10e^{\frac{1}{40}} - 9} = 0.104.$$

That is, for a homogeneous network, the probability of an empty system decreases as the variability of the gate inter-opening times decreases.

Special case 1: Exponential inter-opening intervals

In the case that gate inter-opening times are exponential (μ) while gate j opens independently every exponential (μ_j) time, $\hat{A}(0) = \frac{\mu}{\lambda + \mu}$, $p_j = \frac{\mu_j}{\mu}$ and the result (8) coincides with Equation (8) in [26], namely,

$$P(L = 0) = G_L(0) = \prod_{j=1}^n \frac{\frac{\mu}{\lambda + \mu} \left(\frac{\mu_j}{\mu}\right)}{1 - \frac{\mu}{\lambda + \mu} \left(1 - \frac{\mu_j}{\mu}\right)} = \prod_{j=1}^n \frac{\frac{\mu_j}{\lambda + \mu}}{1 - \frac{\mu - \mu_j}{\lambda + \mu}} = \prod_{j=1}^n \frac{\mu_j}{\lambda + \mu_j}.$$

That is, the system is empty if all gates open sequentially one after the other, with no arrivals occurring between gate-opening instants.

Special case 2: Deterministic inter-opening intervals

When $\alpha \rightarrow \infty$,

$$\lim_{\alpha \rightarrow \infty} E[z^L] = \prod_{j=1}^n \frac{p_j}{e^{\frac{\lambda(1-z)}{\mu}} - (1 - p_j)}. \tag{12}$$

In this case, Equation (11) becomes

$$P(L = 0) = \prod_{j=1}^n \frac{p_j}{e^{\frac{\lambda}{\mu}} - (1 - p_j)}. \tag{13}$$

2.2.2. Uniformly Distributed Inter-Opening Intervals

Suppose the arrival process is Poisson with rate λ and $O \sim U\left[0, \frac{2}{\mu}\right]$. Then,

$$\hat{A}(z) = \frac{\mu}{2\lambda(1-z)} \left(1 - e^{-\frac{2\lambda(1-z)}{\mu}}\right) \quad \text{and} \quad \hat{A}(0) = \frac{\mu}{2\lambda} \left(1 - e^{-\frac{2\lambda}{\mu}}\right). \tag{14}$$

Consequently, substituting Equation (14) in Equation (7) results in

$$G_L(z) = E[z^L] = \prod_{j=1}^n \frac{p_j}{\frac{2\lambda(1-z)}{\mu} \left(1 - e^{-\frac{2\lambda(1-z)}{\mu}}\right)^{-1} - (1 - p_j)} \tag{15}$$

and

$$P(L = 0) = G_L(0) = \prod_{j=1}^n \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1 - p_j)}. \tag{16}$$

2.2.3. The Impact of the Inter-Opening Times on $P(L = 0)$: Deterministic vs. Uniform

Comparing Equation (13) with Equation (16), it follows that $P(L = 0)$ under the uniform distribution is higher than its corresponding value under the deterministic distribution, as $\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} < e^{\frac{\lambda}{\mu}}$.

2.3. Busy Period

Let B denote the network’s busy period, which is the time elapsed from the instant that a particle is admitted to an empty system until the first moment thereafter that the system becomes empty again. Additionally, let Δ_0 be the time elapsed from the instant that a particle is admitted to the first site until the first instant thereafter that (at least) another particle is admitted to that site. As before, T denotes the traversal time of a particle, where T is independent of Δ_0 . Then,

$$B = \begin{cases} T & T < \Delta_0 \\ \Delta_0 + B' & T \geq \Delta_0 \end{cases} \tag{17}$$

where B' is an i.i.d replica of B . Then, the Laplace–Stieltjes transform of B is given by

$$\begin{aligned} \tilde{B}(s) &= E[e^{-sB}] = E[e^{-sT} | T < \Delta_0] P(T < \Delta_0) \\ &+ E[e^{-s(\Delta_0+B')} | T \geq \Delta_0] P(T \geq \Delta_0) \end{aligned}$$

We can define $f_T(t)$ and $g_{\Delta_0}(z)$ as the probability density function of T and of Δ_0 , respectively. We have

$$E[e^{-sB}] = \int_0^\infty f(t)e^{-st} \left(\int_t^\infty g(z)dz \right) dt + E[e^{-sB}] \int_0^\infty f(t) \left(\int_0^t g(z)e^{-sz} dz \right) dt. \tag{18}$$

Then,

$$E[e^{-sB}] = \frac{\int_0^\infty f(t)e^{-st} \left(\int_t^\infty g(z)dz \right) dt}{1 - \int_0^\infty f(t) \left(\int_0^t g(z)e^{-sz} dz \right) dt}. \tag{19}$$

When the arrival is Poisson (λ), i.e., $g(z) = \lambda e^{-\lambda z}$, then

$$E[e^{-sB}] = \frac{\int_0^\infty f(t)e^{-(\lambda+s)t} dt}{1 + \frac{\lambda}{\lambda+s} \int_0^\infty f(t)(e^{-(\lambda+s)t} - 1) dt} = \frac{\tilde{T}(\lambda + s)}{1 + \frac{\lambda}{\lambda+s} (\tilde{T}(\lambda + s) - 1)}. \tag{20}$$

The mean $E[B]$ can be calculated directly by using Equation (17), namely,

$$\begin{aligned} E[B] &= E[T | T < \Delta_0] P(T < \Delta_0) + E[(\Delta_0 + B') | \Delta_0 \leq T] P(\Delta_0 \leq T) \\ &= E[T | T < \Delta_0] P(T < \Delta_0) + (E[\Delta_0 | \Delta_0 \leq T] + E[B' | \Delta_0 \leq T]) P(\Delta_0 \leq T). \end{aligned}$$

Hence, since Δ_0 , T and B' are mutually independent,

$$\begin{aligned} E[B] &= \int_0^\infty t \cdot f(t) \left(\int_t^\infty g(z)dz \right) dt + \int_0^\infty f(t) \left(\int_0^t z \cdot g(z)dz \right) dt \\ &+ E[B] \int_0^\infty f(t) \int_0^t g(z)dz dt \end{aligned}$$

Thus,

$$E[B] = \frac{\int_0^\infty t \cdot f(t) \left(\int_t^\infty g(z)dz \right) dt + \int_0^\infty f(t) \left(\int_0^t z \cdot g(z)dz \right) dt}{1 - \int_0^\infty f(t) \int_0^t g(z)dz dt}. \tag{21}$$

When the arrival process is Poisson with rate λ , Equation (21) becomes

$$\begin{aligned}
 E[B] &= \frac{\int_0^\infty t \cdot f(t) \left(\int_t^\infty \lambda e^{-\lambda z} dz \right) dt + \int_0^\infty f(t) \left(\int_0^t z \cdot \lambda e^{-\lambda z} dz \right) dt}{1 - \int_0^\infty f(t) \int_0^t \lambda e^{-\lambda z} dz dt} \\
 &= \frac{\int_0^\infty t \cdot f(t) e^{-\lambda t} dt + \int_0^\infty f(t) \left(-t \cdot e^{-\lambda t} - \frac{e^{-\lambda t}}{\lambda} + \frac{1}{\lambda} \right) dt}{1 - \int_0^\infty f(t) (-e^{-\lambda t} + 1) dt} \\
 &= \frac{\frac{1}{\lambda} \int_0^\infty f(t) (1 - e^{-\lambda t}) dt}{\int_0^\infty f(t) e^{-\lambda t} dt} = \frac{1}{\lambda} \cdot \frac{1 - \tilde{T}(\lambda)}{\tilde{T}(\lambda)}.
 \end{aligned}
 \tag{22}$$

2.3.1. Gamma-Distributed Inter-Opening Intervals

When the inter-opening time O follows the Gamma distribution $\Gamma(\alpha, \alpha\mu)$ and the arrival process is Poisson (λ), then substituting Equation (3) in both Equations (20) and (22), respectively, leads to

$$E[e^{-sB}] = \frac{\prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda + s}{\alpha\mu}\right)^\alpha - (1 - p_j)}}{1 + \frac{\lambda}{\lambda + s} \left(\prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda + s}{\alpha\mu}\right)^\alpha - (1 - p_j)} - 1 \right)}
 \tag{23}$$

and

$$E[B] = \frac{1}{\lambda} \cdot \left(\prod_{j=1}^n \left(1 + \frac{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - 1}{p_j} \right) - 1 \right)
 \tag{24}$$

We conclude that $E[B](\alpha)$ is a mildly strictly increasing function of α since $\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha$ is a strictly increasing function of α for a given ratio of $\frac{\lambda}{\mu}$, but the increasing rate is mild, as seen below.

Example. Consider a homogeneous system with $n = 10$ sites where $p_j = \frac{1}{n} = \frac{1}{10}$. Suppose that $\lambda = \frac{1}{10}$ and $\mu = 4$. Substituting the above values in Equation (24) yields

$$\begin{aligned}
 E[B](\alpha) &= 10 \left(\prod_{j=1}^{10} \left(10 \left(1 + \frac{1}{40\alpha} \right)^\alpha - 9 \right) - 1 \right) = 10 \left(\left(10 \left(1 + \frac{1}{40\alpha} \right)^\alpha - 9 \right)^{10} - 1 \right) \\
 E[B](1) &= 10 \left(\left(10 \cdot \left(1 + \frac{1}{40} \right) - 9 \right)^{10} - 1 \right) = 81.13 \\
 E[B](2) &= 10 \left(\left(10 \cdot \left(1 + \frac{1}{80} \right)^2 - 9 \right)^{10} - 1 \right) = 84.30 \\
 \lim_{\alpha \rightarrow \infty} E[B](\alpha) &= 10 \left(\left(10e^{\frac{1}{40}} - 9 \right)^{10} - 1 \right) = 85.51
 \end{aligned}$$

Special case 1: Exponential inter-opening intervals

If gate j opens independently every exponential time with parameter μ_j , by substituting Equation (3) in Equation (20), we obtain

$$\begin{aligned}
 E[e^{-sB}] &= \frac{\prod_{j=1}^n \frac{\mu_j}{\mu_j + \lambda + s}}{1 + \frac{\lambda}{\lambda + s} \prod_{j=1}^n \frac{\mu_j}{\mu_j + \lambda + s} - \frac{\lambda}{\lambda + s}} = \frac{(\lambda + s) \prod_{j=1}^n \frac{\mu_j}{\mu_j + \lambda + s}}{s + \lambda \prod_{j=1}^n \frac{\mu_j}{\mu_j + \lambda + s}} \\
 &= \frac{\lambda + s}{s \prod_{j=1}^n \frac{\mu_j + \lambda + s}{\mu_j} + \lambda} = \frac{\lambda + s}{\lambda + s \prod_{j=1}^n \left(1 + \frac{\lambda + s}{\mu_j}\right)}.
 \end{aligned}
 \tag{25}$$

Indeed, Equation (25) coincides with Equation (11) in [26], and Equation (26) below coincides with Equation (10) in [26],

$$E[B] = \frac{1}{\lambda} \left(\prod_{j=1}^n \left(1 + \frac{\lambda}{\mu_j}\right) - 1 \right).
 \tag{26}$$

Special case 2: Deterministic inter-opening intervals

Taking the limit $\alpha \rightarrow \infty$ in Equation (23) gives

$$\begin{aligned}
 \lim_{\alpha \rightarrow \infty} E[e^{-sB}] &= \lim_{\alpha \rightarrow \infty} \frac{\prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda + s}{\alpha \mu}\right)^\alpha - (1 - p_j)}}{1 + \frac{\lambda}{\lambda + s} \left(\prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda + s}{\alpha \mu}\right)^\alpha - (1 - p_j)} - 1 \right)} \\
 &= \frac{\prod_{j=1}^n \frac{p_j}{e^{\frac{\lambda + s}{\mu}} - (1 - p_j)}}{1 + \frac{\lambda}{\lambda + s} \left(\prod_{j=1}^n \frac{p_j}{e^{\frac{\lambda + s}{\mu}} - (1 - p_j)} - 1 \right)},
 \end{aligned}
 \tag{27}$$

while Equation (24) becomes

$$\lim_{\alpha \rightarrow \infty} E[B] = \lim_{\alpha \rightarrow \infty} \frac{1}{\lambda} \cdot \left(\prod_{j=1}^n \frac{\left(1 + \frac{\lambda}{\alpha \mu}\right)^\alpha - (1 - p_j)}{p_j} - 1 \right) = \frac{1}{\lambda} \cdot \left(\prod_{j=1}^n \frac{e^{\frac{\lambda}{\mu}} - (1 - p_j)}{p_j} - 1 \right).
 \tag{28}$$

2.3.2. Uniformly Distributed Inter-Opening Intervals

Substituting Equation (6) in Equation (20) and in Equation (22) gives

$$\begin{aligned}
 E[e^{-sB}] &= \frac{\prod_{j=1}^n \frac{p_j}{\frac{2(\lambda + s)}{\mu} \left(1 - e^{-\frac{2(\lambda + s)}{\mu}}\right)^{-1} - (1 - p_j)}}{1 + \frac{\lambda}{\lambda + s} \left(\prod_{j=1}^n \frac{p_j}{\frac{2(\lambda + s)}{\mu} \left(1 - e^{-\frac{2(\lambda + s)}{\mu}}\right)^{-1} - (1 - p_j)} - 1 \right)},
 \end{aligned}
 \tag{29}$$

and

$$E[B] = \frac{1}{\lambda} \cdot \frac{1 - \prod_{j=1}^n \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1-p_j)}}{\prod_{j=1}^n \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1-p_j)}} = \frac{1}{\lambda} \cdot \left(\prod_{j=1}^n \frac{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1-p_j)}{p_j} - 1 \right). \tag{30}$$

2.3.3. Comparison between the Deterministic and the Uniform Distributions

Comparing Equation (28) to Equation (30), it follows that the mean busy period under the uniform distribution is smaller than its corresponding value under the deterministic distribution, since $\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} < e^{\frac{\lambda}{\mu}}$.

2.4. First Occupied Site

Let I denote the index of the first occupied site such that $P(I = k)$ denotes the probability that the first occupied site in the system is site k . Clearly,

$$P(I = k) = P(X_1 = \dots = X_{k-1} = 0, X_k > 0) = P(X_1 = \dots = X_{k-1} = 0) - P(X_1 = \dots = X_k = 0).$$

$$\text{According to Equation (8), } P(X_1 = \dots = X_k = 0) = P(L_k = 0) = \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)}.$$

Therefore,

$$\begin{aligned} P(I = k) &= P(L_{k-1} = 0) - P(L_k = 0) \\ &= \prod_{j=1}^{k-1} \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)} - \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)} \\ &= \left(\prod_{j=1}^{k-1} \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)} \left(1 - \frac{\hat{A}(0)p_k}{1 - \hat{A}(0)(1-p_k)} \right) \right) \\ &= \frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)}. \end{aligned} \tag{31}$$

Comparing the probabilities of site $k + 1$ and site k to be the first occupied site, yields

$$\frac{P(I = k + 1)}{P(I = k)} = \frac{\frac{1 - \hat{A}(0)}{\hat{A}(0)p_{k+1}} \prod_{j=1}^{k+1} \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)}}{\frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1-p_j)}} = \frac{\hat{A}(0)p_k}{1 - \hat{A}(0)(1-p_{k+1})}.$$

It follows that in the homogeneous case where all p_j are equal, the above ratio is smaller than 1, implying that $P(I = k)$ decreases with k .

2.4.1. Gamma-Distributed Inter-Opening Intervals

In the case that the inter-opening times are Gamma-distributed, substituting Equation (9) in Equation (31) yields

$$P(I = k) = \frac{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - (1-p_j)}. \tag{32}$$

Special case 1: Exponential inter-opening intervals

Compared with the results in [26] where $\hat{A}(0) = \frac{\mu}{\lambda + \mu}$ (Poisson arrival) and $p_j = \frac{\mu_j}{\mu}$, after some algebra, Equation (32) coincides with Equation (14) in [26], namely, $P(I = k) = \frac{\lambda}{\mu_k} \prod_{j=1}^k \frac{\mu_j}{\lambda + \mu_j}$.

Special case 2: Deterministic inter-opening intervals

If the inter-opening times are deterministic ($\alpha \rightarrow \infty$), then

$$P(I = k) = \frac{e^{\frac{\lambda}{\mu}} - 1}{p_k} \prod_{j=1}^k \frac{p_j}{e^{\frac{\lambda}{\mu}} - (1 - p_j)}. \tag{33}$$

2.4.2. Uniformly Distributed Inter-Opening Times

If the inter-opening times are uniformly distributed, then

$$P(I = k) = \frac{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1 - p_j)} \tag{34}$$

2.5. Draining Time

Let D denote the length of time from the moment the arrival process stops right after a gate-opening instant until the system is totally empty. Let R_k be the remaining time from that moment until the particle that resides in site k exits the system. Then, for a general gate inter-opening interval O , the mean draining time satisfies

$$\begin{aligned} E[D] &= \sum_{k=1}^n P(I = k) E[R_k] \\ &= \sum_{k=1}^n \left(\frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} \right) \left(\sum_{j=k}^n \frac{1}{p_j} \right) \cdot E[O] \end{aligned} \tag{35}$$

and

$$\begin{aligned} E[e^{-sD}] &= E[E[e^{-sD} | I]] = P(L = 0) + \sum_{k=1}^n P(I = k) \prod_{j=k}^n \frac{p_j \tilde{O}(s)}{1 - (1 - p_j) \tilde{O}(s)} \\ &= \prod_{j=1}^n \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} \\ &+ \sum_{k=1}^n \left(\frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} \right) \prod_{j=k}^n \frac{p_j \tilde{O}(s)}{1 - (1 - p_j) \tilde{O}(s)}. \end{aligned} \tag{36}$$

2.5.1. Gamma-Distributed Inter-Opening Intervals

In the case that the inter-opening times are Gamma-distributed, substituting Equation (32) in Equation (35) and in Equation (36) yields

$$E[D] = \frac{1}{\mu} \cdot \sum_{k=1}^n \left(\frac{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - (1 - p_j)} \right) \left(\sum_{j=k}^n \frac{1}{p_j} \right) \tag{37}$$

and

$$E[e^{-sD}] = \prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - (1-p_j)} + \sum_{k=1}^n \left(\frac{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\left(1 + \frac{\lambda}{\alpha\mu}\right)^\alpha - (1-p_j)} \right) \prod_{j=k}^n \frac{p_j}{\left(1 + \frac{s}{\alpha\mu}\right)^\alpha - (1-p_j)}. \tag{38}$$

Special case 1: Exponential inter-opening intervals

When $\alpha = 1$, Equation (37) and Equation (38) become, respectively,

$$E[D] = \sum_{k=1}^n \left(\frac{1 - \frac{\mu}{\lambda + \mu}}{\frac{\mu}{\lambda + \mu} \prod_{j=1}^k \frac{\mu_j}{\lambda + \mu_j}} \prod_{j=1}^k \frac{\mu}{\lambda + \mu} \frac{\mu_j}{\mu} \right) \left(\sum_{j=k}^n \frac{1}{p_j} \right) \cdot E[O]$$

$$= \sum_{k=1}^n \left(\frac{\lambda}{\mu_k \prod_{j=1}^k \lambda + \mu_j} \right) \left(\sum_{j=k}^n \frac{1}{\mu_j} \right)$$

and

$$E[e^{-sD}] = \prod_{j=1}^n \frac{p_j}{\left(1 + \frac{\lambda}{\mu}\right) - (1-p_j)} + \sum_{k=1}^n \left(\frac{\left(1 + \frac{\lambda}{\mu}\right) - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\left(1 + \frac{\lambda}{\mu}\right) - (1-p_j)} \right) \prod_{j=k}^n \frac{p_j}{\left(1 + \frac{s}{\mu}\right) - (1-p_j)}$$

$$= \prod_{j=1}^n \frac{\mu p_j}{\lambda + \mu p_j} + \sum_{k=1}^n \left(\frac{\lambda}{p_k \mu \prod_{j=1}^k \lambda + \mu p_j} \right) \prod_{j=k}^n \frac{\mu p_j}{s + \mu p_j},$$

which coincide with Equation (16) and with Equation (17) in [26].

Special case 2: Deterministic inter-opening intervals

Taking the limit $\alpha \rightarrow \infty$ results in

$$\lim_{\alpha \rightarrow \infty} E[D] = \frac{1}{\mu} \cdot \sum_{k=1}^n \left(\frac{e^{\frac{\lambda}{\mu}} - 1}{p_k} \prod_{j=1}^k \frac{p_j}{e^{\frac{\lambda}{\mu}} - (1-p_j)} \right) \left(\sum_{j=k}^n \frac{1}{p_j} \right) \tag{39}$$

and

$$\lim_{\alpha \rightarrow \infty} E[e^{-sD}] = \prod_{j=1}^n \frac{p_j}{e^{\frac{\lambda}{\mu}} - (1-p_j)} + \sum_{k=1}^n \left(\frac{e^{\frac{\lambda}{\mu}} - 1}{p_k} \prod_{j=1}^k \frac{p_j}{e^{\frac{\lambda}{\mu}} - (1-p_j)} \right) \prod_{j=k}^n \frac{p_j}{e^{\frac{s}{\mu}} - (1-p_j)}. \tag{40}$$

2.5.2. Uniformly Distributed Inter-Opening Intervals

Substituting Equation (34) in Equation (35) and in Equation (36), respectively, yields

$$E[D] = \sum_{k=1}^n \left(\frac{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1-p_j)} \right) \left(\sum_{j=k}^n \frac{1}{p_j} \right) \cdot E[O] \tag{41}$$

and

$$\begin{aligned}
 E[e^{-sD}] &= \prod_{j=1}^n \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1 - p_j)} \\
 &+ \sum_{k=1}^n \left(\frac{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - 1}{p_k} \prod_{j=1}^k \frac{p_j}{\frac{2\lambda}{\mu} \left(1 - e^{-\frac{2\lambda}{\mu}}\right)^{-1} - (1 - p_j)} \right) \cdot \quad (42) \\
 &\prod_{j=k}^n \frac{p_j}{\frac{2s}{\mu} \left(1 - e^{-\frac{2s}{\mu}}\right)^{-1} - (1 - p_j)}
 \end{aligned}$$

3. Optimization

Suppose that a G-ASIP network as described above is to be constructed, and suppose that the probabilities p_j of the sites can be determined so as to optimize certain performance measures. In particular, we seek an optimal allocation of the gate-opening probabilities, where $\sum_{j=1}^n p_j = 1$, in order to:

- (i) Minimize mean traversal time, $\left\{ E[T] = E[O] \sum_{j=1}^n \frac{1}{p_j} \right\}$, which is equivalent to minimizing mean load, $\left\{ E[L] = \lambda \cdot E[O] \cdot \sum_{j=1}^n \frac{1}{p_j} \right\}$.
- (ii) Maximize the probability of an empty system, $\left\{ P(L = 0) = \prod_{j=1}^n \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} \right\}$.
- (iii) Minimize load variance $V(L)$ subject to $\sum_{j=1}^n p_j = 1$.
- (iv) Minimize load variance $V(L)$ subject to a given load mean $E[L]$, where $E[L] = \lambda \cdot E[O] \cdot \sum_{j=1}^n \frac{1}{p_j} = C$.

The equivalent cases in (i) involve a convex objective function and a single linear constraint. It readily follows that a homogeneous G-ASIP, where $p_j = \frac{1}{n}, j = 1, 2, \dots, n$, is the optimal solution.

Case (ii): Maximizing $P(L = 0)$ is equivalent to minimizing the convex function $\sum_{j=1}^n \ln \left(1 + \frac{1 - \hat{A}(0)}{\hat{A}(0)p_j} \right)$. Thus, again, the homogeneous G-ASIP is optimal.

Cases (iii) and (iv): By differentiating $G_L(z)$ of Equation (7), we obtain (see Appendix A)

$$\begin{aligned}
 V(L) &= \left(\hat{A}''(1) - 2(\hat{A}'(1))^2 + \hat{A}'(1) \right) \sum_{j=1}^n \frac{1}{p_j} \\
 &+ (\hat{A}'(1))^2 \left(2 \sum_{j=1}^n \frac{1}{p_j^2} + \sum_{j=1}^n \frac{1}{p_j} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{1}{p_k} \right) - \left(\hat{A}'(1) \sum_{j=1}^n \frac{1}{p_j} \right)^2
 \end{aligned}$$

It follows that, as in all previous cases, the homogeneous G-ASIP is optimal.

As a result of the above conclusions, we assume a homogeneous G-ASIP for the following analysis. That is, $p_1 = \dots = p_n = \frac{1}{n}$.

4. Asymptotic Analysis: The Homogeneous Generalized ASIP

In what follows, we extend the work of [26] for ASIP systems to homogeneous G-ASIP systems and study the behavior of the performance measures: (i) traversal time, (ii) overall load, (iii) busy period, (iv) first occupied site, and (v) draining time in the cases of heavy traffic ($\lambda \rightarrow \infty$), large systems ($n \rightarrow \infty$), and the more interesting case of balanced systems. We discuss both means and stochastic limit laws for the above measures.

4.1. Heavy Traffic and Large Systems

In heavy traffic, when $\lambda \rightarrow \infty$, the performance measures $E[L]$, $E[B]$, and $E[D]$ all tend toward infinity, $P(L = 0) \rightarrow 0$, while $P(I = 1) \rightarrow 1$. However, the traversal time is independent of λ , and for a homogeneous system, it is given by $E[T] = n^2 \cdot E[O]$.

For a large system, when $n \rightarrow \infty$, $E[T]$, $E[L]$, $E[B]$, and $E[D]$ all tend toward infinity, and $P(L = 0) \rightarrow 0$. In this case, the first occupied site is effectively the first. That is,

$$\lim_{n \rightarrow \infty} P(I = 1) = \lim_{n \rightarrow \infty} \frac{1 - \hat{A}(0)}{1 - \hat{A}(0)\left(1 - \frac{1}{n}\right)} = 1.$$

This follows since the gate of the first site, which opens with probability $p_1 = \frac{1}{n} \rightarrow 0$, is almost always closed.

4.2. Balanced Systems

Consider a balanced system in a homogeneous generalized ASIP system $\left(p_j = \frac{1}{n}\right)$, where the number of sites tends to infinity ($n \rightarrow \infty$), while the mean sojourn time at a site, $nE[O]$, tends to zero, such that $n^2 \cdot E[O] \rightarrow \tau, \tau < \infty$.

4.2.1. Traversal Time

Clearly,

$$E[T] = n^2 \cdot E[O] \rightarrow \tau.$$

4.2.2. Overall Load and the Probability of an Empty System

By Little’s Law, $E[L] = \lambda \cdot E[T] = \lambda \cdot \tau$.

Substituting $p_j = \frac{1}{n}$ in Equation (7) gives the PGF of the overall load,

$$\begin{aligned} G_L(z) = E[z^L] &= \prod_{j=1}^n \frac{\hat{A}(z)^{\frac{1}{n}}}{1 - \hat{A}(z)\left(1 - \frac{1}{n}\right)} \\ &= \left(\frac{n}{\hat{A}(z)} - (n-1)\right)^{-n} = \left(1 + n\left(\frac{1}{\hat{A}(z)} - 1\right)\right)^{-n}. \end{aligned} \tag{43}$$

The probability that the system is empty is

$$P(L = 0) = G_L(0) = \left(1 + n\left(\frac{1}{\hat{A}(0)} - 1\right)\right)^{-n}. \tag{44}$$

In order to investigate the behavior of the performance measures in balanced systems, it is required to specify the distribution of the inter-opening times. As before, we consider the wide family of Gamma distributions as well as the uniform distributions.

4.2.2.1. Gamma-Distributed Inter-Opening Intervals

If the gate inter-opening interval O is distributed like $\Gamma\left(\alpha, \frac{\alpha n^2}{\tau}\right)$ such that $E[O] = \frac{\tau}{n^2}$ and its LST is $\tilde{\Gamma}\left(\alpha, \frac{\alpha n^2}{\tau}\right)(s) = \left(\frac{\alpha n^2}{s\tau + \alpha n^2}\right)^\alpha$, then, if the arrival process is Poisson with rate λ ,

$$\hat{A}(z) = \left(\frac{\alpha n^2}{\tau\lambda(1-z) + \alpha n^2}\right)^\alpha \text{ and } \hat{A}(0) = \left(\frac{\alpha n^2}{\tau\lambda + \alpha n^2}\right)^\alpha. \tag{45}$$

Substituting Equation (45) in Equation (43) and in Equation (44), respectively, gives

$$G_L(z) = \left(1 + n\left(\left(1 + \frac{\tau\lambda(1-z)}{\alpha n^2}\right)^\alpha - 1\right)\right)^{-n},$$

and

$$P(L = 0) = \left(1 + n\left(\left(1 + \frac{\tau\lambda}{\alpha n^2}\right)^\alpha - 1\right)\right)^{-n}.$$

Special case 1: Exponential inter-opening intervals

When $\alpha = 1$, $G_L(z) = \left(1 + \frac{\tau\lambda(1-z)}{n}\right)^{-n}$, so that

$$\lim_{n \rightarrow \infty} G_L(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{\tau\lambda(1-z)}{n}\right)^{-n} = e^{-\tau\lambda(1-z)}. \tag{46}$$

That is, L is distributed as a Poisson random variable with parameter $\lambda\tau$. Equation (46) coincides with Equation (32) in [26].

The probability that the system is empty is $P(L = 0) = \left(1 + \frac{\tau\lambda}{n}\right)^{-n}$.

If $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} P(L = 0) = \lim_{n \rightarrow \infty} \left(1 + \frac{\tau\lambda}{n}\right)^{-n} = e^{-\tau\lambda},$$

coinciding with Equation (32) in [26].

Special case 2: Deterministic inter-opening intervals

When $\alpha \rightarrow \infty$,

$$\lim_{\alpha \rightarrow \infty} G_L(z) = \lim_{\alpha \rightarrow \infty} \left(1 + n\left(\left(1 + \frac{\tau\lambda(1-z)}{\alpha n^2}\right)^\alpha - 1\right)\right)^{-n} = \left(1 + n\left(e^{\frac{\tau\lambda(1-z)}{n^2}} - 1\right)\right)^{-n}$$

and

$$\lim_{\alpha \rightarrow \infty} P(L = 0) = \lim_{\alpha \rightarrow \infty} \left(1 + n\left(\left(1 + \frac{\tau\lambda}{\alpha n^2}\right)^\alpha - 1\right)\right)^{-n} = \left(1 + n\left(e^{\frac{\tau\lambda}{n^2}} - 1\right)\right)^{-n}.$$

Taking $n \rightarrow \infty$ results in

$$\lim_{n \rightarrow \infty} G_L(z) = \lim_{n \rightarrow \infty} \left(1 + n\left(e^{\frac{\tau\lambda(1-z)}{n^2}} - 1\right)\right)^{-n} = e^{-\tau\lambda(1-z)}$$

and

$$\lim_{n \rightarrow \infty} P(L = 0) = \lim_{n \rightarrow \infty} \left(1 + n\left(e^{\frac{\tau\lambda}{n^2}} - 1\right)\right)^{-n} = e^{-\tau\lambda}.$$

4.2.2.2. Uniformly Distributed Inter-Opening Intervals

If $O \sim U\left[0, \frac{2\tau}{n^2}\right]$ with $E[O] = \frac{\tau}{n^2}$, then, similarly to Equation (17),

$$\hat{A}(z) = \frac{n^2}{2\tau\lambda(1-z)} \left(1 - e^{-\frac{2\tau\lambda(1-z)}{n^2}}\right) \quad \text{and} \quad \hat{A}(0) = \frac{n^2}{2\tau\lambda} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right).$$

Thus,

$$G_L(z) = \left(1 + n \left(\frac{2\tau\lambda(1-z)}{n^2} \left(1 - e^{-\frac{2\tau\lambda(1-z)}{n^2}}\right)^{-1} - 1\right)\right)^{-n},$$

while the probability that the system is empty is

$$P(L = 0) = G_L(0) = \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right)^{-1} - 1\right)\right)^{-n}.$$

If $n \rightarrow \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} G_L(z) &= \left(1 + n \left(\frac{2\tau\lambda(1-z)}{n^2} \left(1 - e^{-\frac{2\tau\lambda(1-z)}{n^2}}\right)^{-1} - 1\right)\right)^{-n} \\ &= \lim_{n \rightarrow \infty} \left(1 + n \left(\frac{1}{1 - \frac{\tau\lambda(1-z)}{n^2}} - 1\right)\right)^{-n} \\ &= \lim_{n \rightarrow \infty} \left(1 + \left(\frac{\tau\lambda(1-z)}{n}\right)\right)^{-n} = e^{-\tau\lambda(1-z)}, \end{aligned}$$

while the probability that the system is empty is

$$\begin{aligned} \lim_{n \rightarrow \infty} P(L = 0) &= \lim_{n \rightarrow \infty} \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right)^{-1} - 1\right)\right)^{-n} \\ &= \lim_{n \rightarrow \infty} \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - \left(1 - \frac{2\tau\lambda}{n^2} + \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} - \dots\right)\right)^{-1} - 1\right)\right)^{-n} \\ &= \lim_{n \rightarrow \infty} \left(1 + n \left(\left(\frac{1}{1 - \frac{\tau\lambda}{n^2}}\right) - 1\right)\right)^{-n} = \lim_{n \rightarrow \infty} \left(1 + n \left(\frac{\tau\lambda}{n^2}\right)\right)^{-n} = e^{-\lambda\tau}. \end{aligned}$$

We conclude that in balanced systems, the limiting behavior of L when n tends to infinity is the same for all three gate inter-opening distributions: exponential, deterministic, and uniform.

4.2.3. Busy Period

Substituting $p_j = \frac{1}{n}$ in Equation (20) gives the Laplace transform of the busy period under a general inter-opening time O :

$$E\left[e^{-sB}\right] = \frac{\left(1 - n \left(1 - \frac{1}{\bar{O}(\lambda + s)}\right)\right)^{-n}}{1 + \frac{\lambda}{\lambda + s} \left(\left(1 - n \left(1 - \frac{1}{\bar{O}(\lambda + s)}\right)\right)^{-n} - 1\right)} \tag{47}$$

while the mean is given by

$$E[B] = \frac{1}{\lambda} \cdot \left(\left(1 - n \left(1 - \frac{1}{\tilde{O}(\lambda)} \right) \right)^{-n} - 1 \right). \tag{48}$$

4.2.3.1. Gamma-Distributed Inter-Opening Intervals

If O is distributed like $\Gamma\left(\alpha, \frac{\alpha n^2}{\tau}\right)$, then

$$\tilde{O}(s) = \left(1 + \frac{s\tau}{\alpha n^2} \right)^{-\alpha}. \tag{49}$$

Substituting Equation (49) in both Equation (47) and in Equation (48), respectively, leads to

$$E[e^{-sB}] = \frac{\left(1 - n \left(1 - \left(1 + \frac{(\lambda+s)\tau}{\alpha n^2} \right)^\alpha \right) \right)^{-n}}{1 + \frac{\lambda}{\lambda+s} \left(\left(1 - n \left(1 - \left(1 + \frac{(\lambda+s)\tau}{\alpha n^2} \right)^\alpha \right) \right)^{-n} - 1 \right)},$$

while the mean is given by

$$E[B] = \frac{1}{\lambda} \cdot \left(\left(1 - n \left(1 - \left(1 + \frac{\lambda\tau}{\alpha n^2} \right)^\alpha \right) \right)^{-n} - 1 \right).$$

Special case 1: Exponential inter-opening intervals

When $\alpha = 1$,

$$E[e^{-sB}] = \frac{\left(1 + \frac{(\lambda+s)\tau}{n} \right)^{-n}}{1 + \frac{\lambda}{\lambda+s} \left(\left(1 + \frac{(\lambda+s)\tau}{n} \right)^{-n} - 1 \right)}$$

and

$$E[B] = \frac{1}{\lambda} \cdot \left(\left(1 + \frac{\lambda\tau}{n} \right)^{-n} - 1 \right).$$

When $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} E[e^{-sB}] = \frac{e^{-(\lambda+s)\tau}}{1 + \frac{\lambda}{\lambda+s} (e^{-(\lambda+s)\tau} - 1)} = \frac{\lambda + s}{(\lambda + s)e^{(\lambda+s)\tau} + \lambda(1 - e^{(\lambda+s)\tau})} = \frac{\lambda + s}{se^{(\lambda+s)\tau} + \lambda}$$

which coincides with Equation (33) in [26]. Then,

$$\lim_{n \rightarrow \infty} E[B] = \frac{1}{\lambda} \cdot (e^{-\lambda\tau} - 1).$$

Special case 2: Deterministic inter-opening time

When $\alpha \rightarrow \infty$,

$$\lim_{\alpha \rightarrow \infty} E[e^{-sB}] = \frac{\left(1 - n \left(1 - e^{\frac{(\lambda+s)\tau}{n^2}} \right) \right)^{-n}}{1 + \frac{\lambda}{\lambda+s} \left(\left(1 - n \left(1 - e^{\frac{(\lambda+s)\tau}{n^2}} \right) \right)^{-n} - 1 \right)}$$

and

$$\lim_{\alpha \rightarrow \infty} E[B] = \frac{1}{\lambda} \cdot \left(\left(1 - n \left(1 - e^{\frac{\lambda\tau}{n^2}} \right) \right)^{-n} - 1 \right).$$

When $n \rightarrow \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} E[e^{-sB}] &= \lim_{n \rightarrow \infty} \frac{\left(1 - n \left(1 - \left(1 + \frac{(\lambda + s)\tau}{n^2}\right)\right)\right)^{-n}}{1 + \frac{\lambda}{\lambda + s} \left(\left(1 - n \left(1 - \left(1 + \frac{(\lambda + s)\tau}{n^2}\right)\right)\right)^{-n} - 1\right)} \\ &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{(\lambda + s)\tau}{n}\right)^{-n}}{1 + \frac{\lambda}{\lambda + s} \left(\left(1 + \frac{(\lambda + s)\tau}{n}\right)^{-n} - 1\right)} \\ &= \frac{e^{-(\lambda + s)\tau}}{1 + \frac{\lambda}{\lambda + s} (e^{-(\lambda + s)\tau} - 1)} = \frac{\lambda + s}{se^{(\lambda + s)\tau} + \lambda} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} E[B] &= \lim_{n \rightarrow \infty} \frac{1}{\lambda} \cdot \left(\left(1 - n \left(1 - e^{\frac{\lambda\tau}{n^2}}\right)\right)^{-n} - 1\right) = \lim_{n \rightarrow \infty} \frac{1}{\lambda} \cdot \left(\left(1 - n \left(1 - \left(1 + \frac{\lambda\tau}{n^2}\right)\right)\right)^{-n} - 1\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\lambda} \cdot \left(\left(1 + \frac{\lambda\tau}{n}\right)^{-n} - 1\right) = \frac{1}{\lambda} \cdot (e^{-\lambda\tau} - 1). \end{aligned}$$

It follows that, when $n \rightarrow \infty$, $E[B]$ is the same when $\alpha = 1$ (exponential) and when $\alpha \rightarrow \infty$ (deterministic). The same holds for $E[e^{-sB}]$, the LST of the busy period.

4.2.3.2. Uniformly Distributed Inter-Opening Intervals

If O is uniformly distributed like $U\left[0, \frac{2\tau}{n^2}\right]$, mean $E[O] = \frac{\tau}{n^2}$, and LST $\tilde{O}(s) = \frac{n^2}{2\tau s} \left(1 - e^{-\frac{2\tau s}{n^2}}\right)$, then

$$E[e^{-sB}] = \frac{\left(1 - n \left(1 - \frac{2\tau(\lambda + s)}{n^2} \left(1 - e^{-\frac{2\tau(\lambda + s)}{n^2}}\right)^{-1}\right)\right)^{-n}}{1 + \frac{\lambda}{\lambda + s} \left(\left(1 - n \left(1 - \frac{2\tau(\lambda + s)}{n^2} \left(1 - e^{-\frac{2\tau(\lambda + s)}{n^2}}\right)^{-1}\right)\right)^{-n} - 1\right)}$$

and

$$E[B] = \frac{1}{\lambda} \cdot \left(\left(1 - n \left(1 - \frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right)^{-1}\right)\right)^{-n} - 1\right).$$

If $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} E[e^{-sB}] &= \lim_{n \rightarrow \infty} \frac{\left(1 - n \left(1 - \frac{2\tau(\lambda + s)}{n^2} \left(1 - e^{-\frac{2\tau(\lambda + s)}{n^2}}\right)^{-1}\right)\right)^{-n}}{1 + \frac{\lambda}{\lambda + s} \left(\left(1 - n \left(1 - \frac{2\tau(\lambda + s)}{n^2} \left(1 - e^{-\frac{2\tau(\lambda + s)}{n^2}}\right)^{-1}\right)\right)^{-n} - 1\right)} \\ &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{\tau(\lambda + s)}{n}\right)^{-n}}{1 + \frac{\lambda}{\lambda + s} \left(\left(1 + \frac{\tau(\lambda + s)}{n}\right)^{-n} - 1\right)} \\ &= \frac{e^{-\tau(\lambda + s)}}{1 + \frac{\lambda}{\lambda + s} (e^{-\tau(\lambda + s)} - 1)} = \frac{(\lambda + s)e^{-\tau(\lambda + s)}}{(\lambda + s) + \lambda e^{-\tau(\lambda + s)} - \lambda} = \frac{\lambda + s}{\lambda + s e^{\tau(\lambda + s)}}, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} E[B] = \frac{1}{\lambda} \cdot \left(\left(1 + \frac{\tau\lambda}{n}\right)^{-n} - 1 \right) = \frac{1}{\lambda} \cdot (e^{-\tau\lambda} - 1).$$

Thus, similarly to L , the distribution of B in balanced systems when n tends to infinity is the same for exponential, deterministic, and uniform distributions.

4.2.4. First Occupied Site

For a general inter-opening time O , substituting $p_j = \frac{1}{n}$ in Equation (31) yields

$$P(I = k) = \frac{1 - \hat{A}(0)}{\hat{A}(0)\frac{1}{n}} \left(\frac{\hat{A}(0)\frac{1}{n}}{1 - \hat{A}(0)\left(1 - \frac{1}{n}\right)} \right)^k = n \left(\frac{1}{\hat{A}(0)} - 1 \right) \left(1 - n \left(1 - \frac{1}{\hat{A}(0)} \right) \right)^{-k}$$

4.2.4.1. Gamma-Distributed Inter-Opening Intervals

In the case that O is Gamma-distributed, using $\hat{A}(0) = \left(\frac{\alpha n^2}{\tau\lambda + \alpha n^2}\right)^\alpha$ results in

$$P(I = k) = n \left(\left(1 + \frac{\tau\lambda}{\alpha n^2}\right)^\alpha - 1 \right) \left(1 - n \left(1 - \left(1 + \frac{\tau\lambda}{\alpha n^2}\right)^\alpha \right) \right)^{-k}.$$

In what follows, we define the scaled first occupied site $\hat{I} = \frac{I}{n}$ and use a Riemann sum, where $\lim_{n \rightarrow \infty} \sum_{i=1}^k f\left(\frac{i}{n}\right) \frac{1}{n} = \int_0^x f(t) dt$ and $x = \frac{k}{n}$.

Special case 1: Exponential inter-opening intervals

$$P(I = k) = \frac{\tau\lambda}{n} \left(1 + \frac{\tau\lambda}{n}\right)^{-k}.$$

When $n \rightarrow \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} P(\hat{I} \leq x) &= \lim_{n \rightarrow \infty} \sum_{i=1}^k \frac{\tau\lambda}{n} \left(1 + \frac{\tau\lambda}{n}\right)^{-i} \\ &= \tau\lambda \lim_{n \rightarrow \infty} \sum_{i=1}^k \left(\left(1 + \frac{\tau\lambda}{n}\right)^n \right)^{-\frac{i}{n}} \frac{1}{n} = \tau\lambda \int_0^x e^{-\tau\lambda \cdot t} dt = 1 - e^{-\lambda\tau x}. \end{aligned}$$

Special case 2: Deterministic inter-opening intervals

When $\alpha \rightarrow \infty$

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} P(I = k) &= \lim_{\alpha \rightarrow \infty} n \left(\left(1 + \frac{\tau\lambda}{\alpha n^2} \right)^\alpha - 1 \right) \left(1 - n \left(1 - \left(1 + \frac{\tau\lambda}{\alpha n^2} \right)^\alpha \right) \right)^{-k} \\ &= n \left(e^{\frac{\tau\lambda}{n^2}} - 1 \right) \left(1 - n \left(1 - e^{\frac{\tau\lambda}{n^2}} \right) \right)^{-k}. \end{aligned}$$

If $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} P(\hat{I} \leq x) = \lim_{n \rightarrow \infty} \sum_{i=1}^k \frac{\tau\lambda}{n} \left(1 + \frac{\tau\lambda}{n} \right)^{-i} = 1 - e^{-\lambda\tau x},$$

Again, it follows that for both exponential and deterministic distributions, the scaled first occupied site follows an exponential distribution with parameter $(\lambda\tau)$.

4.2.4.2. Uniformly Distributed Inter-Opening Intervals

Substituting $\hat{A}(0) = \frac{n^2}{2\tau\lambda} \left(1 - e^{-\frac{2\tau\lambda}{n^2}} \right)$ in $P(I = k)$ in Section 4.2.4.1 leads to

$$P(I = k) = n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}} \right)^{-1} - 1 \right) \left(1 - n \left(1 - \frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}} \right)^{-1} \right) \right)^{-k}.$$

When $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} P(I = k) &= \lim_{n \rightarrow \infty} n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}} \right)^{-1} - 1 \right) \left(1 - n \left(1 - \frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}} \right)^{-1} \right) \right)^{-k} \\ &= \lim_{n \rightarrow \infty} n \left(\frac{2\tau\lambda}{n^2} \left(1 - \left(1 - \frac{2\tau\lambda}{n^2} + \frac{2(\tau\lambda)^2}{n^4} \right) \right)^{-1} - 1 \right) \cdot \\ &\quad \left(1 - n \left(1 - \frac{2\tau\lambda}{n^2} \left(1 - \left(1 - \frac{2\tau\lambda}{n^2} + \frac{2(\tau\lambda)^2}{n^4} \right) \right)^{-1} \right) \right)^{-k} \\ &= \lim_{n \rightarrow \infty} n \left(\left(1 - \frac{\tau\lambda}{n^2} \right)^{-1} - 1 \right) \left(1 - n \left(1 - \left(1 - \frac{\tau\lambda}{n^2} \right)^{-1} \right) \right)^{-k} = \lim_{n \rightarrow \infty} \frac{\tau\lambda}{n} \left(1 + \frac{\tau\lambda}{n} \right)^{-k}. \end{aligned}$$

We conclude that when $n \rightarrow \infty$ the exponential, deterministic and uniform distributions yield the same exponential-type result.

4.2.5. Draining Time

Substituting $p_j = \frac{1}{n}$ in Equation (36) yields

$$\begin{aligned} E[e^{-sD}] &= \left(1 + n \left(\frac{1}{\hat{A}(0)} - 1 \right) \right)^{-n} \\ &\quad + \sum_{k=1}^n \left(n \left(\frac{1}{\hat{A}(0)} - 1 \right) \left(1 + n \left(\frac{1}{\hat{A}(0)} - 1 \right) \right)^{-k} \right) \left(1 + n \left(\frac{1}{\hat{O}(s)} - 1 \right) \right)^{k-n-1} \end{aligned} \tag{50}$$

4.2.5.1. Gamma-Distributed Inter-Opening Intervals

Substituting $\hat{A}(0) = \left(\frac{\alpha n^2}{\tau\lambda + \alpha n^2} \right)^\alpha$ in Equation (50) gives

$$\begin{aligned} E[e^{-sD}] &= \left(1 + n \left(\left(1 + \frac{\tau\lambda}{\alpha n^2} \right)^\alpha - 1 \right) \right)^{-n} \\ &\quad + \sum_{k=1}^n \left(n \left(\left(1 + \frac{\tau\lambda}{\alpha n^2} \right)^\alpha - 1 \right) \left(1 + n \left(\left(1 + \frac{\tau\lambda}{\alpha n^2} \right)^\alpha - 1 \right) \right)^{-k} \right) \left(1 + n \left(\left(1 + \frac{s\tau}{\alpha n^2} \right)^\alpha - 1 \right) \right)^{k-n-1}. \end{aligned}$$

Special case 1: Exponential inter-opening intervals

When $\alpha = 1$,

$$\begin{aligned} E[e^{-sD}] &= \left(1 + \frac{\tau\lambda}{n}\right)^{-n} + \frac{\tau\lambda}{n} \sum_{k=1}^n \left(1 + \frac{\tau\lambda}{n}\right)^{-k} \left(1 + \frac{s\tau}{n}\right)^{k-n-1} \\ &= \left(1 + \frac{\tau\lambda}{n}\right)^{-n} + \frac{\tau\lambda}{n} \sum_{k=1}^n \left(\frac{n}{n+\tau\lambda}\right)^k \left(\frac{n}{n+s\tau}\right)^{n-k+1} \\ &= \left(1 + \frac{\tau\lambda}{n}\right)^{-n} + \frac{\tau\lambda}{n} \frac{n}{n+\tau\lambda} \frac{n}{n+s\tau} \frac{\left(\frac{n}{n+\tau\lambda}\right)^n - \left(\frac{n}{n+s\tau}\right)^n}{\frac{n}{n+\tau\lambda} - \frac{n}{n+s\tau}} \\ &= \left(1 + \frac{\tau\lambda}{n}\right)^{-n} + \lambda \frac{\left(1 + \frac{\tau\lambda}{n}\right)^{-n} - \left(1 + \frac{s\tau}{n}\right)^{-n}}{s-\lambda}. \end{aligned}$$

When $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} E[e^{-sD}] = e^{-\tau\lambda} + \lambda \frac{e^{-\tau\lambda} - e^{-\tau s}}{s - \lambda} = \frac{se^{-\tau\lambda} - \lambda e^{-\tau s}}{s - \lambda},$$

which coincides with Equation (36) in [26].

By differentiation,

$$E[D] = -\frac{d}{ds} E[e^{-sD}]_{s=0} = -\frac{-\lambda e^{-\tau\lambda} - \lambda^2\tau + \lambda}{\lambda^2} = \tau - \frac{1 - e^{-\tau\lambda}}{\lambda}.$$

Special case 2: Deterministic inter-opening intervals

When $\alpha \rightarrow \infty$,

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} E[e^{-sD}] &= \left(1 + n \left(e^{\frac{\tau\lambda}{n^2}} - 1\right)\right)^{-n} \\ &\quad + \sum_{k=1}^n \left(n \left(e^{\frac{\tau\lambda}{n^2}} - 1\right) \left(1 + n \left(e^{\frac{\tau\lambda}{n^2}} - 1\right)\right)^{-k}\right) \left(1 + n \left(e^{\frac{\tau s}{n^2}} - 1\right)\right)^{k-n-1} \end{aligned}$$

When $n \rightarrow \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} E[e^{-sD}] &= \lim_{n \rightarrow \infty} \left(1 + n \left(1 + \frac{\tau\lambda}{n^2} + \dots - 1\right)\right)^{-n} \\ &\quad + \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(n \left(1 + \frac{\tau\lambda}{n^2} + \dots - 1\right) \left(1 + n \left(1 + \frac{\tau\lambda}{n^2} + \dots - 1\right)\right)^{-k}\right) \left(1 + n \left(1 + \frac{\tau s}{n^2} + \dots - 1\right)\right)^{k-n-1} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{\tau\lambda}{n}\right)^{-n} + \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{\tau\lambda}{n} \left(1 + \frac{\tau\lambda}{n}\right)^{-k}\right) \left(1 + \frac{\tau s}{n}\right)^{k-n-1} = \frac{se^{-\tau\lambda} - \lambda e^{-\tau s}}{s-\lambda}. \end{aligned}$$

4.2.5.2. Uniformly Distributed Inter-Opening Intervals

$$\begin{aligned} E[e^{-sD}] &= \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right)^{-1} - 1\right)\right)^{-n} \\ &\quad + \sum_{k=1}^n \left(n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right)^{-1} - 1\right) \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - e^{-\frac{2\tau\lambda}{n^2}}\right)^{-1} - 1\right)\right)^{-k}\right) \\ &\quad \left(1 + n \left(\frac{2\tau s}{n^2} \left(1 - e^{-\frac{2\tau s}{n^2}}\right)^{-1} - 1\right)\right)^{k-n-1} \end{aligned}$$

When $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} E[e^{-sD}] &= \lim_{n \rightarrow \infty} \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - \left(1 - \frac{2\tau\lambda}{n^2} + \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} - \dots \right) \right)^{-1} - 1 \right) \right)^{-n} \\ &+ \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(n \left(\frac{2\tau\lambda}{n^2} \left(1 - \left(1 - \frac{2\tau\lambda}{n^2} + \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} - \dots \right) \right)^{-1} - 1 \right) \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(1 - \left(1 - \frac{2\tau\lambda}{n^2} + \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} - \dots \right) \right)^{-1} - 1 \right) \right)^{-k} \right) \\ &\cdot \left(1 + n \left(\frac{2\tau s}{n^2} \left(1 - \left(1 - \frac{2\tau s}{n^2} + \frac{\left(\frac{2\tau s}{n^2}\right)^2}{2} - \dots \right) \right)^{-1} - 1 \right) \right)^{k-n-1} \end{aligned}$$

which gives

$$\begin{aligned} \lim_{n \rightarrow \infty} E[e^{-sD}] &= \lim_{n \rightarrow \infty} \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(\frac{2\tau\lambda}{n^2} - \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} \right)^{-1} - 1 \right) \right)^{-n} \\ &+ \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(n \left(\frac{2\tau\lambda}{n^2} \left(\frac{2\tau\lambda}{n^2} - \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} \right)^{-1} - 1 \right) \left(1 + n \left(\frac{2\tau\lambda}{n^2} \left(\frac{2\tau\lambda}{n^2} - \frac{\left(\frac{2\tau\lambda}{n^2}\right)^2}{2} \right)^{-1} - 1 \right) \right)^{-k} \right) \\ &\cdot \left(1 + n \left(\frac{2\tau s}{n^2} \left(\frac{2\tau s}{n^2} - \frac{\left(\frac{2\tau s}{n^2}\right)^2}{2} \right)^{-1} - 1 \right) \right)^{k-n-1} . \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} E[e^{-sD}] = \lim_{n \rightarrow \infty} \left(1 + \frac{\tau\lambda}{n} \right)^{-n} + \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{\tau\lambda}{n} \left(1 + \frac{\tau\lambda}{n} \right)^{-k} \right) \left(1 + \frac{\tau s}{n} \right)^{k-n-1} = \frac{se^{-\tau\lambda} - \lambda e^{-\tau s}}{s - \lambda}$$

Again, the exponential, deterministic, and uniform distributions admit the same result.

5. Summary

This paper extends the research on generalized ASIP systems where a general renewal process controls intervals between gate-opening instants. Expressions are derived for the Laplace–Stieltjes transform, or PGF, as well as the means of various performance measures. Explicitly, it is shown that:

- (i) The LST of the traversal time T is given by

$$\tilde{T}(s) = \prod_{j=1}^n \frac{p_j \tilde{O}(s)}{1 - (1 - p_j) \tilde{O}(s)},$$

where $\tilde{O}(s)$ is the LST of the intervals between successive gate openings, and p_j is the probability that gate j opens at a gate-opening instant. The mean traversal time is given by

$$E[T] = E[O] \sum_{j=1}^n \frac{1}{p_j}.$$

- (ii) When the arrival process is Poisson (λ), the LST of the busy period B is given by

$$E[e^{-sB}] = \frac{\tilde{T}(\lambda + s)}{1 + \frac{\lambda}{\lambda + s} (\tilde{T}(\lambda + s) - 1)},$$

with mean $E[B] = \frac{1 - \tilde{T}(\lambda)}{\lambda \cdot \tilde{T}(\lambda)}$.

(iii) The probability that the first occupied site is site k is given by

$$P(I = k) = \frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)},$$

where $\hat{A}(z)$ is the PGF of the number of arrivals at the first site during an inter-opening interval O .

(iv) The LST of the draining time D is given by

$$E[e^{-sD}] = \prod_{j=1}^n \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} + \sum_{k=1}^n \left(\frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} \right) \prod_{j=k}^n \frac{p_j \tilde{O}(s)}{1 - (1 - p_j)\tilde{O}(s)}$$

while the mean draining time is

$$E[D] = \sum_{k=1}^n \left(\frac{1 - \hat{A}(0)}{\hat{A}(0)p_k} \prod_{j=1}^k \frac{\hat{A}(0)p_j}{1 - \hat{A}(0)(1 - p_j)} \right) \left(\sum_{j=k}^n \frac{1}{p_j} \right) \cdot E[O].$$

Explicit results are obtained for the family of Gamma-distributed inter-opening intervals (that span the range between the exponential and the deterministic probability distributions), as well as for the uniform distribution. It is further shown that a homogeneous system, where at gate-opening instants gate j opens with probability $p_j = \frac{1}{n}$, is optimal with respect to (i) minimizing mean traversal time, (ii) minimizing the system's load, (iii) maximizing the probability of an empty system, (iv) minimizing mean draining time, and (v) minimizing the load variance. The asymptotic cases of (i) heavy traffic, (ii) large systems, and (iii) balanced systems are further investigated for a homogeneous system. It is further shown that in the intriguing case of balanced systems when the gate inter-opening times distributions are either exponential, deterministic, or uniform, in all three cases:

The system's load PGF, $G_L(z)$, satisfies

$$\lim_{n \rightarrow \infty} G_L(z) = e^{-\tau\lambda(1-z)},$$

where L denotes the total load of the system, and τ is the mean traversal time in a balanced system. The probability that the system is empty is $\lim_{n \rightarrow \infty} P(L = 0) = e^{-\tau\lambda}$.

The LST of the busy period B is given by

$$\lim_{n \rightarrow \infty} E[e^{-sB}] = \frac{\lambda + s}{\lambda + se^{(\lambda+s)\tau}},$$

with mean $\lim_{n \rightarrow \infty} E[B] = \frac{1}{\lambda} \cdot (e^{-\tau\lambda} - 1)$.

The LST of the draining time D obeys

$$\lim_{n \rightarrow \infty} E[e^{-sD}] = \frac{se^{-\tau\lambda} - \lambda e^{-\tau s}}{s - \lambda},$$

and its mean satisfies $\lim_{n \rightarrow \infty} E[D] = \tau - \frac{1 - e^{-\tau\lambda}}{\lambda}$.

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Appendix A

From Equation (7),

$$G_L(z) = \prod_{j=1}^n \frac{\hat{A}(z)p_j}{1 - \hat{A}(z)(1 - p_j)}.$$

The load variance is calculated by differentiating $G_L(z)$ twice. We get

$$G_L'(z) = \sum_{j=1}^n \frac{\hat{A}'(z)p_j}{(1 - \hat{A}(z)(1 - p_j))^2} \prod_{\substack{k=1 \\ k \neq j}}^n \frac{\hat{A}(z)p_k}{1 - \hat{A}(z)(1 - p_k)}.$$

Differentiating again,

$$\begin{aligned} G_L''(z) &= \sum_{j=1}^n p_j \frac{\hat{A}''(z)(1 - \hat{A}(z)(1 - p_j)) + 2(\hat{A}'(z))^2(1 - p_j)}{(1 - \hat{A}(z)(1 - p_j))^3} \prod_{\substack{k=1 \\ k \neq j}}^n \frac{\hat{A}(z)p_k}{1 - \hat{A}(z)(1 - p_k)} \\ &+ \sum_{j=1}^n \frac{\hat{A}'(z)p_j}{(1 - \hat{A}(z)(1 - p_j))^2} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{\hat{A}'(z)p_k}{(1 - \hat{A}(z)(1 - p_k))^2} \prod_{\substack{i=1 \\ i \neq k, j}}^n \frac{\hat{A}(z)p_i}{1 - \hat{A}(z)(1 - p_i)}. \end{aligned}$$

Substituting $z = 1$,

$$\begin{aligned} G_L''(1) &= E[L^2] - E[L] = \sum_{j=1}^n \frac{\hat{A}''(1)p_j + 2(\hat{A}'(1))^2(1 - p_j)}{p_j^2} + \sum_{j=1}^n \frac{\hat{A}'(1)}{p_j} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{\hat{A}'(1)}{p_k} \\ &= (\hat{A}''(1) - 2(\hat{A}'(1))^2) \sum_{j=1}^n \frac{1}{p_j} + (\hat{A}'(1))^2 \left(2 \sum_{j=1}^n \frac{1}{p_j^2} + \sum_{j=1}^n \frac{1}{p_j} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{1}{p_k} \right), \end{aligned}$$

where $E[L] = \hat{A}'(1) \sum_{j=1}^n \frac{1}{p_j}$,

$$E[L^2] = (\hat{A}''(1) - 2(\hat{A}'(1))^2 + \hat{A}'(1)) \sum_{j=1}^n \frac{1}{p_j} + (\hat{A}'(1))^2 \left(2 \sum_{j=1}^n \frac{1}{p_j^2} + \sum_{j=1}^n \frac{1}{p_j} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{1}{p_k} \right).$$

Therefore,

$$\begin{aligned} V(L) &= E[L^2] - E^2[L] = (\hat{A}''(1) - 2(\hat{A}'(1))^2 + \hat{A}'(1)) \sum_{j=1}^n \frac{1}{p_j} \\ &+ (\hat{A}'(1))^2 \left(2 \sum_{j=1}^n \frac{1}{p_j^2} + \sum_{j=1}^n \frac{1}{p_j} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{1}{p_k} \right) - \left(\hat{A}'(1) \sum_{j=1}^n \frac{1}{p_j} \right)^2. \end{aligned}$$

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